

IDEAL THEORY AND PRÜFER DOMAINS

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INTEGER-VALUED POLYNOMIALS II

The primary purpose of this lecture is to show that $\text{Int}(\mathbb{Z})$ is a Prüfer domain. After doing so, we prove that $\text{Int}(\mathbb{Z})$ satisfies the Skolem property: for any polynomials $f_1, \dots, f_k \in \text{Int}(\mathbb{Z})$ the fact that $\gcd(f_1(n), \dots, f_k(n)) = 1$ for every $n \in \mathbb{Z}$ implies that $c_1(x)f_1(x) + \dots + c_k(x)f_k(x) = 1$ for some $g_1, \dots, g_k \in \text{Int}(\mathbb{Z})$.

Theorem 1. *The ring $\text{Int}(\mathbb{Z})$ is a Prüfer domain.*

Proof. Let I be a nonzero finitely generated ideal of $\text{Int}(\mathbb{Z})$, and write $I = (f_1, \dots, f_k)$ for some $f_1, \dots, f_k \in I$. Take $J = (\text{Int}(\mathbb{Z}) : I)$. Since $IJ \subseteq R$, proving that I is invertible amounts to arguing that IJ is not contained in any maximal ideal of $\text{Int}(R)$, that is, $IJ \not\subseteq M_{p,\alpha} = \{f \in \text{Int}(\mathbb{Z}) : f(\alpha) \in p\mathbb{Z}_p\}$ for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$. To do this, fix $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$, and then set

$$(0.1) \quad n := \inf\{v_p(f(\alpha)) : f \in I\}.$$

Since $I\mathbb{Q}[x]$ is an ideal of $\mathbb{Q}[x]$ it is principal and, therefore, we can take $g(x) \in \mathbb{Q}[x]$ such that $I\mathbb{Q}[x] = g(x)\mathbb{Q}[x]$. We can actually assume that $g(x) \in \mathbb{Z}[x]$, and we do so. For each $i \in \llbracket 1, k \rrbracket$, there is a nonzero $d_i \in \mathbb{Z}$ such that $d_i f_i(x) \in g(x)\mathbb{Z}[x]$, and so after setting $d = d_1 \cdots d_k$, we obtain that $dI \subseteq g(x)\text{Int}(\mathbb{Z})$. As a result, $(d/g)I$ is a finitely generated ideal of $\text{Int}(\mathbb{Z})$, which is invertible if and only if I is invertible. Observe, in addition, that $(d/g)I$ contains a nonzero constant polynomial. Thus, after replacing I by $(d/g)I$, we can further assume that I contains a nonzero constant polynomial. This implies that $n \in \mathbb{N}_0$.

For each $i \in \llbracket 1, k \rrbracket$, the continuity of f_i guarantees the existence of a clopen subset U_i of α such that $v_p(f_i(x)) \geq n$ for all $x \in U_i$. Now consider the clopen subset $U := \bigcap_{i=1}^k U_i$ of \mathbb{Z}_p containing α . For all $f \in I$, we can easily see that $v_p(f(x)) \geq n$ for all $x \in U$ and $v_p(f(x)) \geq 0$ for all $x \in \mathbb{Z}_p \setminus U$. Now we can use Stone-Weierstrass Theorem to produce an $h \in \text{Int}(\mathbb{Z})$ with $v_p(h(x)) = 0$ for all $x \in U$ and $v_p(h(x)) = n$ for all $x \in \mathbb{Z}_p \setminus U$. Set $h_0 := h/p^n$, and note that $h_0 \in J$ because $fh_0 \in \text{Int}(\mathbb{Z})$. By n belongs to \mathbb{N}_0 , there exists $f_0 \in I$ such that $v_p(f_0(\alpha)) = n$. As a consequence, even though $f_0 h_0 \in IJ$, the equality $v_p(f_0(\alpha)h_0(\alpha)) = 0$ implies that $f_0 h_0 \notin M_{p,\alpha}$. Hence $IJ \not\subseteq M_{p,\alpha}$ for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$, and so I is an invertible ideal of $\text{Int}(\mathbb{Z})$. We can now conclude that $\text{Int}(\mathbb{Z})$ is a Prüfer domain. \square

Since we have seen before that $\text{Int}(\mathbb{Z})$ has Krull dimension two, the fact that $\text{Int}(\mathbb{Z})$ is a Prüfer domain implies that it is not Noetherian. We record this observation here as a corollary.

Corollary 2. *The ring $\text{Int}(\mathbb{Z})$ is not Noetherian.*

Let us show explicitly that none of the maximal ideals of $\text{int}(\mathbb{Z})$ are finitely generated.

Proposition 3. *Every nonzero prime ideal of $\text{Int}(\mathbb{Z})$ is non-finitely generated.*

Proof. Let us argue first that every finitely generated proper ideal of $\text{Int}(\mathbb{Z})$ is contained in infinitely many maximal ideals. To do this, fix a nonzero finitely generated proper ideal $I = (f_1, \dots, f_k)$, where $f_1, \dots, f_k \in \text{Int}(\mathbb{Z})$ are nonzero. Then there exist $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$ such that I is contained in the maximal ideal $M_{p,\alpha}$. Then $v_p(f_i(\alpha)) \geq 1$ and, by continuity, there exists a clopen subset of \mathbb{Z}_p containing α such that $v_p(f(\beta)) \geq 1$ for all $f \in I$ and $\beta \in U$. Therefore I is contained in $M_{p,\beta}$. Since $M_{p,\beta} \neq M_{p,\beta'}$ for any $\beta, \beta' \in \mathbb{Z}_p$ with $\beta \neq \beta'$, the ideal I is contained in infinitely many maximal ideal.

As a result, the maximal ideals of $\text{Int}(\mathbb{Z})$ cannot be finitely generated. It suffices to argue now that none of the prime ideals $P_q = \text{Int}(\mathbb{Z}) \cap q(x)\mathbb{Q}[x]$, where $q(x)$ is an irreducible of $\mathbb{Q}[x]$, is finitely generated. We have proved before that if P_q is contained in a maximal ideal $M_{p,\alpha}$ for some $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$, then $q(\alpha) = 0$. Since the $M_{p,\alpha}$'s are the only maximal ideals of $\text{Int}(\mathbb{Z})$, the ideal P_q is contained in only finitely many maximal ideals and, therefore, it cannot be finitely generated by the argument established in the previous paragraph. \square

The Skolem Property. For every ideal I of $\text{Int}(\mathbb{Z})$ and for every $n \in \mathbb{Z}$, set $I(n) := \{f(n) : f \in I\}$. It is clear that $I(n)$ is an ideal of \mathbb{Z} for every $n \in \mathbb{Z}$. It turns out that if I is a finitely generated ideal of $\text{Int}(\mathbb{Z})$ satisfying that $I(n) = \mathbb{Z}$ for every $n \in \mathbb{Z}$, then $I = \text{Int}(\mathbb{Z})$. This property satisfied by $\text{Int}(\mathbb{Z})$ is referred to as the *Skolem property*, as it was first proved by T. Skolem [3].

Proposition 4. *If I is a finitely generated ideal of $\text{Int}(\mathbb{Z})$ satisfying that $I(n) = \mathbb{Z}$ for every $n \in \mathbb{Z}$, then $I = \text{Int}(\mathbb{Z})$*

Proof. Let I be a finitely generated ideal of $\text{Int}(\mathbb{Z})$ such that $I(n) = \mathbb{Z}$ for every $n \in \mathbb{Z}$. Assume, towards a contradiction, that I is a proper ideal. Then I is contained in one of the maximal ideals $M_{p,\alpha}$ for some $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$. Since I is generated by finitely many polynomials in $M_{p,\alpha}$, which are continuous and have p -valuation at least 1, there exists a clopen subset U of \mathbb{Z}_p containing α such that $f(\beta) \in p\mathbb{Z}_p$ for all $f \in I$ and $\beta \in U$, that is, $I \subseteq M_{p,\beta}$ for all $\beta \in U$. Since \mathbb{Z} is dense in \mathbb{Z}_p , we can choose $n \in \mathbb{Z} \cap U$ such that $I \subseteq M_{p,n}$. This means that $I(n) \subseteq p\mathbb{Z}$, which is a contradiction. \square

It is worth to emphasize that, unlike $\text{Int}(\mathbb{Z})$, the ring $\mathbb{Z}[x]$ does not satisfy the Skolem property. The following example was given by Skolem in [3].

Example 5. Consider the polynomials $f(x) = 3$ and $g(x) = x^2 + 1$ in $\mathbb{Z}[x]$, and let I be the ideal generated by f and g in $\mathbb{Z}[x]$. Since -1 is not a quadratic residue modulo 3, we see that $\gcd(3, n^2 + 1) = 1$ for every $n \in \mathbb{Z}$. Now suppose, by way of contradiction, that $I = \mathbb{Z}[x]$ and write $3a(x) + (x^2 + 1)b(x) = 1$ for some $a(x), b(x) \in \mathbb{Z}[x]$. After evaluating at the imaginary unit i , we obtain that $1/3 = a(i)$, which means that $1/3$ is a Gaussian integer, a contradiction.

EXERCISES

Exercise 1. *Without using the description of $\text{Spec}(\text{Int}(\mathbb{Z}))$ we have established in the previous lecture, show that the ideal $\{f \in \text{Int}(\mathbb{Z}) : f(0) \in 2\mathbb{Z}\}$ of $\text{Int}(\mathbb{Z})$ is not finitely generated.*

Exercise 2. *[Strong Skolem Property] Let I and J be two finitely generated ideals of $\text{Int}(\mathbb{Z})$ such that $I(n) = J(n)$ for every $n \in \mathbb{Z}$. Prove that $I = J$.*

REFERENCES

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